ON THE THEORY OF QUASILINEAR EQUATIONS

(K TEORII KVAZILINEINYKH URAVNENII)

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S.V. FAL'KOVICH
(Saratov)

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A system of differential equations of the form

$$A_1 \frac{\partial u}{\partial x} + B_1 \frac{\partial u}{\partial y} + C_1 \frac{\partial v}{\partial x} + D_1 \frac{\partial v}{\partial y} + E_1 = 0$$
$$A_2 \frac{\partial u}{\partial x} + B_2 \frac{\partial u}{\partial y} + C_2 \frac{\partial v}{\partial x} + D_2 \frac{\partial v}{\partial y} + E_2 = 0 \qquad (1)$$

is often encountered in problems of the mechanics of continuous media. Examples are plane and axially-symmetric irrotational steady gas flow, unsteady gas flow dependent on one space coordinate, the plane problem of limiting equilibrium of friable media, the diffusion of long waves into rivers and channels, and other problems.

We shall assume, as happens in the above problems, that A_1, A_2, B_1, ..., D_2 are known functions of u and v; and that E_1 and E_2 are functions of u, v, x and y.

We assume that all of these functions are continuous and have as many derivatives as are required. We shall consider the case where $ac - b^2 < 0$; then the system (1) is of hyperbolic type. Here

$$a = [AC], \qquad 2b = [AD] + [BC], \qquad c = [BD]$$

employing the shortened notation of Courant and Friedrichs [1]

$$[XY] = X_1 Y_2 - X_2 Y_1$$

The equation $ax^2 - 2bx + c = 0$ will have two real roots in this case, $\chi_1 \neq \chi_2$. For solution of boundary problems with parametric characteristic variables (α, β), the system (1) leads to the canonical form (for details see Courant and Friedrichs [1]):

$$\frac{\partial y}{\partial \alpha} - \chi_1 \frac{\partial x}{\partial \alpha} = 0, \qquad T \frac{\partial u}{\partial \alpha} + (a\chi_1 - S) \frac{\partial v}{\partial \alpha} + K \frac{\partial y}{\partial \alpha} - H \frac{\partial x}{\partial \alpha} = 0 \qquad (2)$$

$$\frac{\partial y}{\partial \beta} - \chi_2 \frac{\partial x}{\partial \beta} = 0, \qquad T\frac{\partial u}{\partial \beta} + (a\chi_2 - S)\frac{\partial v}{\partial \beta} + K\frac{\partial y}{\partial \beta} - H\frac{\partial x}{\partial \beta} = 0$$

Here

$$T = [AB], \qquad S = [BC], \qquad K = [AE], \qquad H = [BE]$$

As is well known, each solution of the system (2) satisfies the original system (1) if the Jacobian $\partial(x, y)/\partial(\alpha, \beta)$ does not reduce to zero in the plane of the variables (α, β).

We consider the case where at a certain point in the xy plane

$$J = \frac{\partial(u, v)}{\partial(x, y)} = 0$$

If one of the derivatives $\partial J/\partial x$ or $\partial J/\partial y$ is not zero at this point, then $J = 0$ along the entire curve γ in the xy plane. The solution of Equation (1) and $u(x, y)$, $v(x, y)$ is determined in the uv plane, generally speaking, as a certain curve Γ – an image of the curve γ.

We consider the solution to the Cauchy problem in the case where boundary values are given on the curve γ. Let the equation of the curve γ be $x = x(s)$, $y = y(s)$, and the values of the unknown functions on the curve γ be $u = u(s)$, $v = v(s)$. We get, upon differentiation:

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}, \qquad \frac{dv}{ds} = \frac{\partial v}{\partial x}\frac{dx}{ds} + \frac{\partial v}{\partial y}\frac{dy}{ds} \tag{3}$$

The derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ along the curve γ may be found from (1) and (3). Upon substitution of these values into the equation

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 0 \tag{4}$$

we obtain

$$T du^2 + 2U du\, dv + V dv^2 + (K dy - H dx)\, du + (M dy - L dx)\, dv = 0 \tag{5}$$

Here

$$V = [CD], \qquad 2U = [AD] - [BC], \qquad M = [DE], \qquad L = [CE]$$

We find, upon multiplication by the second of Equations (2), that

$$T^2 du^2 + 2UT\, du\, dv + TV\, dv^2 + 2(U\, dv + T\, du)(K\, dy - H\, dx) + (K\, dy - H\, dx)^2 = 0 \tag{6}$$

If Equations (1) are homogeneous, $E_1 = E_2 = 0$, then $K = H = M = L = 0$, and Equation (5) coincides with Equation (6) for the characteristic; the curve Γ and the curve γ, consequently, appear as characteristics.

If Equations (1) are not homogeneous, i.e. if $E_1 \neq 0$, $E_2 \neq 0$, then the values of K, L, M and H are different from zero, and Equation (5) will not in general coincide with Equation (6).

Consequently, in this case the curves Γ *and are not characteristics.*

The last observation is essential, since the difficulties in the solution of boundary problems when the system (1) is nonhomogeneous may not be surmounted by applying the Khristianovich method [2] of multisheeted surfaces, because the smooth edge of such a surface (curve Γ) will not coincide with a characteristic and may only be determined if the solution itself is known [3].

Incorrect statements are encountered in the literature to the effect that the curve $J = 0$ is a characteristic when the system (1) is nonhomogeneous (see Sokolovskii's book [4] in which there are a number of solutions based upon such statements).

BIBLIOGRAPHY

1. Courant, R. and Friedrichs, K., *Sverkhzvukovoe techenie i udarnye volny (Supersonic Flow and Shock Waves)*. IIL, 1950.

2. Khristianovich, S.A., *Neustanovivsheesia dvizhenie v kanalakh i rekakh. Nekotorye novye voprosy mekhaniki sploshnoi sredy (Nonstationary Motion in Channels and Rivers. Certain New Problems in the Mechanics of Continuous Media)*. Izd-vo Akad. Nauk SSSR, 1938.

3. Ryzhov, O.S., O techeniiakh v okrestnosti poverkhnosti perekhoda v soplakh Lavalia (On the flows at openings in Laval nozzles). *PMM* Vol. 22, No. 4, 1958.

4. Sokolovskii, V.V., *Statika sypuchei sredy (Statics of Friable [Granular] Media)*. Fizmatgiz, 1960

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